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Pages 65

Celestial Mechanics

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Celestial mechanics can be defined as the branch of astronomy which describes mathematically the motions of celestial bodies. It started with the discovery of the law of universal gravitation by Sir Isaac Newton and with the publication of his <u>Principia</u> (1687).

Two-body Problem

Using Newton's expression for the force of attraction between two bodies

$$\vec{F} = f \frac{m_1 m_2}{r^2}$$

the problem of the determination of the heliocentric position vector of a planet, or of a comet, with the mass m relative to the sun, having the mass M, can be reduced to the integration of the differential equation of two body problem:

$$\frac{d^2 \vec{r}}{dt^2} = - f \frac{M+m}{r^3} \vec{r}$$

The integration of this equation shows that the trajectory is a conic section situated in a fixed plane. The integrals of Eq. (1) contain six independent constants of integration - the elements of the motion. They are: the mean anomaly at the epoch - M_o, the argument of the perihelion ω, the longitude of the ascending node Ω, the inclination of the

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orbital plane i, the eccentricity e, and the semimajor axis a of the orbit (or the mean daily motion - n). Sometimes the angle $\pi = \omega + \Omega$ the longitude of the perigee, is used instead of ω and the mean longitude at the epoch $\varepsilon = M + \pi$ is used instead of M. For computational reasons it is more convenient to introduce the unit vector $\vec{P}$ directed toward the perigee and $\vec{R}$, the unit vector normal to the orbital plane, instead of ω, Ω, i. The theory of the two-body problem can also be based upon two vectorial integrals: the area integral

$$\vec{r} \times \frac{d\vec{r}}{dt} = \vec{c} = \text{const.},$$

and the Laplacian integral

$$\vec{c} \times \frac{d\vec{r}}{dt} + f(M+m)\left(\frac{\vec{r}}{r} + e\,\vec{P}\right) = 0$$

The position vector in the planetary motion can be expressed in one of two forms:

$$\vec{r} = \vec{P}\, r\cos v + \vec{Q}\, r\sin v = \vec{P}\, a(\cos E - e) + \vec{Q}\, a\sqrt{1-e^2}\,\sin E$$

where v is the angle between $\vec{r}$ and $\vec{P}$. It is the true anomaly. The angle E is called the eccentric anomaly.

To the last equation must be added the classical Kepler's integral:

$$E - e\sin E = M,$$

where $M = M_0 + n(t - t_0)$ is the mean anomaly.

Many-body Problem

The two-body problem can serve, however, only as a first approximation. In the planetary system, consisting of bodies with the masses m_i and with the heliocentric position vectors $\vec{r}_i$ $(i=1,2,\dots n)$ the motion is not governed by Eq. (1) but, because of the mutual attraction of bodies, by more complicated equations:

$$\frac{d^2\vec{r}_i}{dt^2} = -f\,\frac{M+m_i}{r_i^3}\,\vec{r}_i + \sum_{j\neq i} f m_j \left(\frac{\vec{r}_j-\vec{r}_i}{|\vec{r}_j-\vec{r}_i|^3} - \frac{\vec{r}_j}{r_j^3}\right) \tag{3}$$

which can be written in the form

$$\frac{d^2\vec{r}_i}{dt^2} = -f\,\frac{(M+m_i)}{r_i^3}\,\vec{r}_i + \operatorname{grad} R_i \tag{4}$$

where

$$R_i = \sum_{j\neq i} f m_j \left(\frac{1}{|\vec{r}_j-\vec{r}_i|} - \frac{\vec{r}_i\cdot\vec{r}_j}{r_j^3}\right) \tag{5}$$

and it is called the disturbing function of the i-th planet. In the case of only two ~~bodies~~ planets it is customary to write Eq. (4) and (5) as

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\mu^2}{r^3}\,\vec{r} + \operatorname{grad} R \tag{6}$$

where

$$R = f m' \left(\frac{1}{\rho} - \frac{\vec{r}\cdot\vec{r}\,'}{r'^3}\right) \tag{7}$$

where $\rho = |\vec{r}\,' - \vec{r}|$, $\mu^2 = f\,(M+m)$

The body with the mass $\underline{m}$ and the position vector $\underline{\vec{r}}$ is termed as "disturbed" and the body with the mass m' and the position vector $\vec{r}\,'$

is termed as "disturbing". The differential Eq. (6) together with the corresponding equation for the body m' defines the celebrated three-body problem. The closed form complete solution of this problem ~~cannot~~ were ~~be~~ not found. If equations of motion are given relative to an inertial system, then the classical scalar integrals do exist. In the vectorial form, these integrals are: the integrals of motion of the centers of masses

$$\sum_i m_i \frac{d\vec{r}_i}{dt} = \vec{A} = \text{const.}$$

$$\sum_i m_i \vec{r}_i = \vec{A}t + \vec{B}, \qquad \vec{B} = \text{const.}$$

which says that the center of masses moves rectilinearly with the constant velocity, and the area integral

$$\sum_i m_i \vec{r}_i \times \frac{d\vec{r}_i}{dt} = \vec{C} = \text{const.}$$

and the integral of energy

$$\sum_i m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 = \sum_{\substack{i,j \\ i \neq j}} f \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|} + h$$

These integrals can be used for the purpose of the reduction of the three-body problem to lesser degrees of freedom (from 18 to 12, then to 8, finally to 6), but any attempt to find any new integrals failed. H. Bruns (1887) has shown that any integral algebraic in the coordinates and the velocities must be an algebraic combination of ten classical integrals. This important result was extended by H. Poincaré (1892), who has shown that any integral represented by a uniform and transcendental function of the coordinates and the velocities is again a combination of ten classical integrals. Finally, Painlevé (1898) has shown that the integrals which are algebraic in velocities only and different from the ten known integrals cannot exist. Thus the unknown integrals in the three-body problem must be multivalued and transcendental functions. It is no surprise that they have not been found.

Planetary Theories

If the system consists of the sun and only one planet, then the planet moves around the sun in an ellipse of constant shape and position. The elements of this ellipse can be deduced from the position and the velocity vector of the planet at any particular moment.

In the planetary system, the motion of each planet is disturbed by the attraction of the other members of the system,and its trajectory is no longer an ellipse of constant shape. The position and the velocity vector at a given moment define only an instantaneous, osculating, ellipse and the real motion of the planet can be understood as a continuous set of transitions from one osculating ellipse to the other.

If the disturbing action of other planets would suddenly disappear, the planet would continue to move in its osculating ellipse. The dependence of the position and of the velocity vectors upon time and upon the elements c_i $(i=1,2,\ldots 6)$ of the osculating ellipse must be the same as in the two-body problem. In other words, if the position and the velocity of the planet are

$$\vec{r} = \vec{r}\,(t\,;\,c_1, c_2, \ldots, c_6),$$
$$\vec{v} = \vec{v}\,(t, c_1, c_2, \ldots, c_6),$$

then we must have

$$\frac{\partial \vec{r}}{\partial t} = \vec{v},$$

$$\frac{\partial \vec{v}}{\partial t} = -\frac{\mu^2}{r^3}\vec{r}$$

where

$$\mu^2 = f\,(M+m)$$

Let us first consider the case of only two planets. Let us designate their masses by m and m', their position vectors by $\vec{r}$ and $\vec{r}\,'$ and the velocity vectors by $\vec{v}$ and $\vec{v}\,'$, respectively. The differential equation of the heliocentric disturbed motion of the planet m is

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\mu^2}{r^3}\vec{r} + fm'\left(\frac{\vec{r}\,' - \vec{r}}{\rho^3} - \frac{\vec{r}\,'}{r'^3}\right) \qquad (10)$$

where

$$\rho = |\,\vec{r}\,' - \vec{r}\,|$$

The second term in Eq. (10) represents the disturbing acceleration. It consists of two terms. The first term has as its source the direct attraction of m by m' . The second, "indirect," term is a "reflection" of motion of m' around the sun. Its presence is caused by the transformation from the inertial system of coordinates to the system of coordinates associated with the sun.

Introducing the "disturbing function"

$$R = fm'\left(\frac{1}{\rho} - \frac{\vec{r}\cdot\vec{r}'}{r'^3}\right), \tag{11}$$

we can write Eq. (10) as

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\mu^2}{r^3}\vec{r} + \operatorname{grad} R. \tag{12}$$

In the case of several disturbing planets, the disturbing functions of the i-th planet is

$$R_i = \sum_{j \neq i} f m_j \left(\frac{1}{\rho_{ij}} - \frac{\vec{r}_i\cdot\vec{r}_j}{r_j^3}\right),$$

where the sum is taken over all the disturbing planets and the differential equation of motion of the i-th planet takes the form

$$\frac{d^2\vec{r}_i}{dt^2} = -\frac{\mu_i^2}{r_i^3}\vec{r}_i + \operatorname{grad}_{\vec{r}_i} R_i \tag{13}$$

Taking into account

$$\frac{d\vec{r}}{dt} = \vec{v} = \frac{\partial \vec{r}}{\partial t} + \sum_{k=1}^{6} \frac{dc_k}{dt}\frac{\partial \vec{r}}{\partial c_k},$$

$$\frac{d\vec{v}}{dt} = -\frac{\kappa^2 \vec{r}}{r^3} + \operatorname{grad} R = \frac{\partial \vec{v}}{\partial t} + \sum_{k=1}^{6} \frac{dc_k}{dt}\frac{\partial \vec{v}}{\partial c_k}$$

and Eqs. (8) and (9) we obtain the conditions of the osculation in the form

$$\sum_{k=1}^{6} \frac{dc_k}{dt}\frac{\partial \vec{r}}{\partial c_k} = 0$$

$$\sum_{k=1}^{6} \frac{dc_k}{dt}\frac{\partial \vec{v}}{\partial c_k} = \operatorname{grad} R$$

From these two equations we obtain

$$\sum_{k=1}^{6} \frac{dc_k}{dt}\,[c_j, c_k] = \frac{\partial R}{\partial c_j}, \quad (i,j = 1, 2, \dots 6), \tag{14}$$

where

$$[c_j, c_k] = \frac{\partial \vec{r}}{\partial c_j}\cdot\frac{\partial \vec{v}}{\partial c_k} - \frac{\partial \vec{r}}{\partial c_k}\cdot\frac{\partial \vec{v}}{\partial c_j}$$

are the Lagrangian brackets. In the Eq. (14) is contained all the theory of osculation. From ~~the~~ these equations the differential equations for the variations of the osculating elements in time can be deduced. J. L. Lagrange (1736-1813) has obtained the following differential equations for the variations of the classical elliptic elements:

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \varepsilon},$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\,\sin i}\frac{\partial R}{\partial i},$$

$$\frac{d\varpi}{dt} = \frac{\tan\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e},$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varpi} - \sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varepsilon},$$

$$\frac{di}{dt} = -\frac{1}{na^2\sqrt{1-e^2}\,\sin i}\frac{\partial R}{\partial \Omega} - \frac{\tan\frac{i}{2}}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \varepsilon}\right),$$

$$\frac{d\varepsilon}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\tan\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e}.$$

The use of the elliptical elements is not always convenient because the eccentricity and the inclination enter as divisors into the differential equation. We have considerable numerical difficulties if these divisors are small, especially in the case of nearly circular orbits. This difficulty can be circumvented by employing some other sets of elements, such as, for example, the canonical elements of Delaunay

$$L = \mu\sqrt{a}, \qquad l = M$$

$$G = L\sqrt{1-e^2}, \qquad g = \omega$$

$$H = G\cos i, \qquad h = \Omega,$$

in the theory of the moon, or the non-singular set of elements of Poincare:

$$L = \mu\sqrt{a} \qquad \lambda = l + g + h$$
$$\xi_1 = +\sqrt{2(L-G)}\cos(g+h) \qquad \eta_1 = +\sqrt{2(G-H)}\cos h$$
$$\xi_2 = -\sqrt{2(L-G)}\sin(g+h) \qquad \eta_2 = -\sqrt{2(G-H)}\sin h$$

The simplest way to determine the osculating elements as functions of time would be to integrate the differential equations numerically. However, taking into account the fact that the motions of the planets are nearly periodic, we come to the conclusion that we can determine the perturbations by developing them into a trigonometric series with several arguments. Such an approach has served as a foundation for planetary theories since the time of Laplace.

The expansion in powers of the inclination is performed first. It requires the developments of the form

$$(1+\alpha^2-2\alpha\cos\varphi)^{-s} = \frac{1}{2}\sum_{-\infty}^{+\infty} b_s^{(i)}\cos i\varphi$$

where φ is the difference of the mean longitudes of both planets. The coefficients $b_s^{(i)}$ were introduced into Celestial Mechanics by Laplace and bear his name.

After the development in inclinations is done, the development in the eccentricities is obtained in terms of Newcomb's differential operators as applied to the Laplacian coefficients. There are tables giving the Laplacian coefficients and their derivatives as functions of $\alpha = a/a'$. At the present time, however, the development of Newcomb's operators, the computation of Laplacian coefficients and their derivatives is done on electronic machines directly in each particular case. The final decomposition of the disturbing function has the form

$$R = \frac{fm'}{a'}\sum A(\alpha)\, e^p e'^{p'} \tan^q\frac{i}{2}\tan^{q'}\frac{i'}{2}\cos D$$

where the argument D is a linear combination of the angular elements with the coefficients which are integers, positive, negative, or zero,

$$D = i\ell + i'\ell' + k\pi + k'\pi' + j(\Omega - \Omega')$$

and $\underline{A}$ is a function of $\underline{a}$ and $\underline{a}'$ of the order -1. If the canonical elements of Poincaré are being used, then the development of $\underline{R}$ has the form

$$R = f m' \sum B(L, L') M \,{}^{\sin}_{\cos}\, (i\lambda + i'\lambda')$$

where $\underline{M}$ are polynomials in the canonical elements ξ and η of both planets. The part of $\underline{R}$ which is independent from the mean anomalies is called the secular disturbing function. In the process of integration, it produces the terms having time as a factor. In the first approximation we consider the elements $\underline{g}$, $\underline{g}'$, $\underline{h}$, $\underline{h}'$, $\underline{e}$, $\underline{e}'$, $\underline{i}$, $\underline{i}'$, or the elements ξ, η and $\underline{L}$ of the disturbed and of the disturbing bodies as constants and make use of the Lagrangian equations for variation of constants. The perturbations are obtained by the integration of Fourier series similar to Eq. (15) with the sine or cosine terms. As a result of integration, the perturbations of the first-order in $\underline{m}$ and $\underline{m}'$ of each element ζ, excepting $\underline{a}$ or $\underline{L}$, have the form:

$$\delta\zeta = \nu t + \sum \frac{K}{in + i'n'} \,{}^{\sin}_{\cos}\, D$$

and for $\underline{a}$ or $\underline{L}$;

$$\delta a \text{ or } \delta L = \sum \frac{K}{in + i'n'} \cos D.$$

In other words the perturbations of all the elements, excepting a or L, consist of secular and periodic terms. The first-order perturbations of a or L contain the periodic terms only (the theorem of Laplace). This result is significant because it give us a first approximation of the idea about the general behavior and stability of the solar system. The itegration of the periodic terms introduces divisors $in + i'n'$, where i and i' are integers. They can be positive or negative. In the perturbations of the mean anomaly, the integration is performed twice, and as a consequence, the squares of the small divisors appear. In the case when n and n' are nearly commensurable, some of these divisors are very small. The period of the corresponding term (the critical term) is very large. The critical terms will also have large amplitudes in the expressions for perturbations, especially in the perturbations of the mean anomaly. The classical examples of the long-period effects are the great inequalities in the mean longitudes of Jupiter, with the amplitude of 1196", and of Saturn, with the amplitude of 2908". They have a period of nearly 900 years and are caused by the commensurability 5/2 of the mean motions of these planets. These two inequalities were discovered by Laplace.

After the first approximation is completed, the higher-order approximations. can be obtained either by developing the perturbations in powers of the disturbing masses or by means of iteration. Each new approximation introduces the mutual actions of more planets into the problem, and consequently more arguments, some of which might be the critical ones. It introduces also the secular terms of higher orders. In addition, the mixed terms, consisting of periodic terms with some power of time as a factor, also appear. After the kth approximation the general term in the planetary perturbations has the form:

$$A \mu^{k} t^{\alpha} N^{-\delta} e^{p} e'^{p'} \tan \frac{i}{2} \tan \frac{i'}{2} \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} D$$

where μ is of the order of the disturbing masses, the argument D is linear with respect to time and has the form:

$$D = (jn + j'n')t + h$$

and N is a small divisor associated with a critical argument. For purely periodic terms $\alpha = 0$, for purely secular terms $|j| + |j'| = 0$, for mixed terms we have $\alpha \neq 0$ $|i| + |i'| \neq 0$. We have to distinguish between the order, the degree, the rank, and the class of a term. The order is the exponent of μ. If α, p, p', q, q' and s are given, then the smaller the order, the smaller the influence of the term over a given interval of time. The degree is the sum $p + p' + q + q'$. Its introduction is justified by the smallness of the eccentricities and of the inclinations. The rank is $k - \alpha$. The terms with the small rank can have great significance even if the order is large, because the term originally small can increase with time. The class is defined as $k - (1/2)(\alpha + \delta)$. The purely secular terms in the first approximation have the order one, the rank zero, and the class 1/2. The long-period terms in the first approximation have the order one, the rank one and the order 1/2. This classification of terms serves as a basis for Poincaré's deep investigations about the analytical structure of the solutions in the planetary theory. The important theorems about the rank are:

(1) The perturbations in the canonical elements ξ, η, λ and L can have only a positive rank.

(2) The rank of the mixed terms cannot be smaller than one.

(3) The perturbations of L do not contain terms of the rank zero.

The fact that there are no terms of the negative rank in any approximation is significant, because it gives a general idea about the speed with which the secular and the mixed terms increase and about the "danger" they represent over a long interval of time. The statement about the absence of the terms of the zero rank in a or L can be taken as an extension of the theorem of Laplace about the stability of the semi-major axes. The terms of the rank zero are always purely secular and they are of primary importance in all investigations concerning the behavior of the planetary system over an extremely long interval of time, say of several hundred thousand years. Next in importance over the very long intervals of time are the terms of the class 1/2, which are either the purely secular or the long periodic. Their influence on the perturbations of higher orders is regulated by the theorem of Poincaré about the class: in the development of all perturbations there are no termsof the negative class and in perturbations of the canonical elements ξ, η, λ the class of terms cannot be smaller than 1/2.

The presence of the secular, of the mixed and of the critical arguments limits the interval of applicability of a planetary theory. For our time the accuracy of the prediction can be brought into agreement with the accuracy of observations. A somewhat lesser accuracy can be achieved for the interval covering the historical period of mankind. An accuracy satisfactory enough to check or to try to explain some astronomical events in the distant past can be achieved. The theorems about ranks and classes provide us with important qualitative information about the behavior of a planetary theory over a long interval of time. However, this information is not of any great service in discussing the problem of stability of the solar system as a whole. In the problem of stability, the terms of the rank zero and of the class 1/2 are still of primary importance; however their treatment must go along different lines than in the standard planetary theories. We consider this problem in the next section.

A different approach to the planetary theory was taken by Hansen. Hansen's basic idea consists of the introduction of a fictitious planet, moving in accordance with Kepler's laws in an ellipse of the constant shape located in the osculating orbit plane. The position of the real planet is determined by its deviation from the position of the fictitious planet in time and space. The position vectors $\vec{r}$ of the real planet at the time t and the position vector $\vec{\bar{r}}$ of the fictitious planet at the time z have the same directions and we can set

$$\vec{r}(t) = (1+\nu)\,\vec{\bar{r}}(z),$$

and in particular

$$r = (1+\nu)\bar{r}$$

The quantities $\nu\bar{r}$ and $\delta z = z - t$ are interpreted as the perturbations of the radius vector and of time. If the mean motion of the fictitious planet is designated by n_o, then its mean anomaly is

$$n_o z + c_o = (n_o t + c_o) + n_o \delta z$$

δz describes the angular deviation of the real planet from the fictitious planet. All the angular perturbations in the orbital plane are combined into one angle $n_o \delta z$ the perturbations of the mean anomaly. The main charm of Hansen's theory is that ν and $n_o \delta z$ can be determined by means of one single function W. The effect of the change of the position of the osculating orbit plane is also taken into account. The orbit plane considered as a rigid body, rotates around the instantaneous axis of rotation, whose direction coincides with the position vector $\vec{r}$ of the real planet. This rotation is caused by the component of the disturbing force normal to the osculating orbit plane and its main perturbative effect is the elevation of the satellite above the initial position of the orbital plane. The method of solution is one of successive approximations. Hansen applied his method to the determination of the perturbations of minor planets and to the development of the theory of Jupiter and Saturn. Later Hill developed the theories of Jupiter and of Saturn using the same principle. Recently, Clemence developed a highly accurate Hansen-type theory of Mars.

In the actual computation of ephemerides we need not so much the osculating elements as the disturbed coordinates. Such a direct approach to determination of the perturbations was taken by Laplace, Newcomb, Hansen and Encke, and recent times by Brouwer, Davis, Danby, Musen and Carpenter. In the method of Laplace, as modified by Newcomb, the perturbations of the log r and of the true orbital longitude of the planet are determined. The instantaneous position of the orbital plane is determined by the perturbations of i and Ω. After these perturbations are found, the determination of the disturbed polar heliocentric coordinates does not present any difficulty.

In the theories of four inner planets, Newcomb made use also of his method of expansion of the disturbing function in terms of symbolic differential operators. Recently Sharaf has applied the method of Laplace-Newcomb to the development of the theory of Pluto.

A new approach to the problem was recently made by Brouwer, who developed the theory of the perturbations in the rectangular coordinates. Let $\delta\vec{r}$, $\delta\vec{r}\,'$, be the deviations of the position vectors from their undisturbed elliptic values, $\vec{r}$ and $\vec{r}\,'$, respectively; $\delta\vec{r}$ satisfies the differential equation of the form

$$\frac{d^2\delta\vec{r}}{dt^2} + \frac{\mu^2}{r^3}\left(\delta\vec{r} - \frac{3}{r^3}\vec{r}\,\vec{r}\cdot\delta\vec{r}\right) \qquad (16)$$

$$= \vec{F}\left(\vec{r}+\delta\vec{r},\ \vec{r}\,'+\delta\vec{r}\,'\right),$$

where $\vec{F}$ is the modified "disturbing acceleration". The undisturbed position vector is the function of time and of six elements: $c_1, c_2, \ldots$ $\ldots c_6$. For $\vec{F} = 0$ the solution of Eq. (16) is

$$\delta \vec{r} = \sum_{i=1}^{6} \frac{\partial \vec{r}}{\partial c_i} \delta c_i .$$

Brouwer makes use of the canonical elements and obtains δc_i for $\vec{F} \neq 0$ by the method of variation of constants. Instead of canonical elements any type of elements can be considered and in the actual computations the elliptic elements are even preferable, as it was shown by Davis. The determination of $\delta\vec{r}$ and $\delta\vec{r}\,'$ in effect can be reduced to the solution of the integral equations of the form

$$\delta \vec{r} = \int \Gamma(t,\tau) \cdot \vec{F} \, dt$$

$$\delta \vec{r}\,' = \int \Gamma'(t,\tau) \cdot \vec{F} \, dt$$

by the method of successive approximation. Hansen's device was used by Brouwer; in other words, after an approximation is completed time τ is replaced by t. Numerical harmonic analysis can be used for the decomposition of $\vec{F}$ into Fourier series. The form of the matrix Γ was determined by Davis. We present here the idea of Brouwer's method with slight changes in notation, by resorting to the terminology of matrix and vector calculus. Danby developed a different form of the theory based on the use of the matrizant. Musen and Carpenter used the decomposition of $\delta\vec{r}$ along $\vec{r}$, $\vec{v}$, and $\vec{R}$ and write

$$\delta\vec{r} = A\vec{r} + B\vec{v} + C\vec{R}$$

and form the differential equations for the determination of A, B, and C. We can write

$$\vec{F}(\vec{r}+\delta\vec{r}, \vec{r}'+\delta\vec{r}') = \exp(\delta\vec{r}\cdot\nabla + \delta\vec{r}'\cdot\nabla')\,\vec{F}(\vec{r},\vec{r}'), \tag{17}$$

where ∇ and ∇' are the gradient operators with respect to $\vec{r}$ and $\vec{r}'$.

Setting

$$\delta\vec{r} = \delta_1\vec{r} + \delta_2\vec{r} + \cdots\cdots$$
$$\delta\vec{r}' = \delta_1\vec{r}' + \delta_2\vec{r}' + \cdots$$

where $\delta_k\vec{r}$, $\delta_k\vec{r}'$ are of the k-order with respect to planetary masses. We can decompose the Taylor displacement operator in the right side of Eq. (17) into a series of operators of Faa de Bruno which are polynomials in $\delta_k\vec{r}$, $\delta_k\vec{r}'$ $(k=1,2,\ldots\ldots)$. With the help of these operators we deduce the decomposition

$$\vec{F}(\vec{r}+\delta\vec{r}, \vec{r}'+\delta\vec{r}') = \vec{F}_1(\vec{r},\vec{r}') + \vec{F}_2(\vec{r},\vec{r}') + \cdots\cdots$$

and reduce the problem to the integration of a set of the differential equations determining the terms of different orders in A, B, C.

Secular Perturbations

Laplace's theorem states that the semimajor axis does not have any secular perturbations of the first order. Poisson has proved that the second-order perturbations do not introduce any secular effects, but that the mixed terms will appear. Haretu has proved that in the third approximation there is a very small secular term of the second rank. Can some conclusion be made about the instability of the solar system in general? Any definite statement on this subject would be too hasty. The appearance of this secular term could probably be ascribed rather to the imperfection of the planetary theories; under any circumstances, the secular term will not have any appreciable influence for several millions of years. The secular terms in other elements, e, i, π, Ω have a greater influence on the behavior of the planetary system over a interval of time, say, of the order of 2×10^6 years. In the linearized form, the differential equations for the secular perturbations led Lagrange to the trigonometrical form of solution, constituting the combination of the periodic terms with periods lying between 57×10^3 and 2×10^6 years. The form of integrals and of the periods in higher approximation were indicated in the works of Poincaré and Hagihara. However, the appearance of extremely small divisors in higher approximations causes considerable numerical difficulties in application of higher orders of theory. On the basis of the linearized differential equations for the secular perturbations, Hirayama and Brouwer discovered families of minor planets. The members of each family, according to Hirayama, have the common origin and represent the fragments of one body. The secular effects play a primary role in the evolution of the planetary orbits and of the eccentric orbits of the artificial satellite. We can predict the planetary effects, but we are not able to observe their full effects because their periods are too large.

In the motion of the artificial satellites we can observe these effects directly. They are caused by the action of the moon. The effective way of determining the secular perturbations of the eccentric orbits is based on the method of Gauss. Gauss removed the short-period effects from the disturbing function by the process of averaging over the orbits of both bodies. Sets of formulas were developed by several mathematicians to perform this averaging numerically. The investigations, by covering the intervals of 2×10^5 years, show that the periodic oscillations in the elements of minor planets are not extremely large, but there are very large changes in the elements of artificial satellites moving originally in the eccentric orbits. The eccentricity, for example, can oscillate between 0.2 and 0.9. Lidov recently made investigations using a different approach. Hamid also came to the same conclusions. Gaussian method cannot be used over the interval of $\sim 2 \times 10^5$ years to investigate the behavior of the planetary orbit, and it cannot be used in the case of cometary orbit over the intervals of the order of 1×10^4 to 2×10^4 years. For the artificial satellites the upper limit seems to be about 20 years. All these estimates of the intervals of the validity of the Gaussian theory are rather conservative, but all attempts to extend the applicability of the existing methods of celestial mechanics to the intervals comparable to the time of existence of the planetary system can be considered as futile. For such intervals of time, the perturbations of the rank zero definitely are insufficient.

Theories of Lunar Motion

The modern theories of the moon are obtained by the development of the perturbations of the coordinates or of the elements into multiple Fourier series. If all known effects are taken into account, then the arguments in these series are the power series with respect to time. If only the solar gravitational effects are considered and, in addition, the motion of the sun is taken to be elliptic, then the arguments become the linear functions of time. We can obtain the solution of the lunar problem in the form of the periodic series because the small divisors in the lunar case are much less troublesome than in the planetary case and in the process of integration no excessively big terms are produced.

The mathematical investigations in the lunar theory started with the publication of Newton's Principia (1687). Newton was able to explain all the periodic inequalities in the motion of the moon which were known in his time from observations, as well as the secular advancement of the lunar perigee and the regression of the node due to the gravitational action of the sun. He also pointed to the existence of some periodic inequalities in the motion of the moon which were not known previously.

The geometrical form into which Newton puts his results makes the reading very difficult, however, from the standpoint of the modern reader.

It seems Clairaut (1765) was the first who made an attempt to create an analytical theory of the moon. He established the differential equations of the motion of the moon with the true longitude as the independent variable. The method of solution is based on the application of successive approximations. The first approximation to the trajectory of the moon is a rotating ellipse. In applying this approximation, Clairaut found that the theoretical value of the motion of the perigee represents only a half of the observed value. As a remedy, Clairaut made an attempt to improve the law of gravitation by adding a term proportional to the inverse cube of the distance. However, after the completion of the second approximation he deduced for the motion of the perigee a value more accurate and in a better agreement with the value observed. D'Alembert (1768, 1773) added more periodic terms and, more important, established that each argument in the periodic series of the lunar theory is a linear combination with the integral coefficients of the four basic arguments.

In the modern lunar theory, the basic arguments are designated by $\underline{l}$, $\underline{l}'$, $\underline{F}$, and $\underline{D}$, where $\underline{l}$ is the mean anomaly of the moon, $\underline{l}'$ the mean anomaly of the sun, $\underline{F}$ the mean angular distance of the moon from the ascending node of its orbit and $\underline{D}$ the mean angular distance of the moon from the sun.

The next important step was done by Euler. He has developed two lunar theories. His second theory (1772) is especially important because its basic idea, the use of the rotating system of coordinates, lies in the foundation of the modern lunar theory by Hill and Brown.

Euler guessed correctly the analytical form of the solution, but he made an error in using the observed values of the motions of the node and of the perigee, instead of deducing them theoretically. As a result, the coefficients in the series representing the solution are distorted.

Three lunar theories are holding the field at the present time: those of Delaunay, Hansen and Hill-Brown. The foundation of the theory of Delaunay is the variations of constants in the canonical form. The expressions of Delaunay canonical variables L, G, H, l, g, h, in terms of the osculating elliptic elements are

$$L = \sqrt{\mu a}\,, \qquad G = \sqrt{\mu a(1-e^2)}\,, \quad H = \sqrt{\mu a(1-e^2)}\,\cos i\,, \tag{18}$$

$$l = \text{the mean anomaly}\,, \qquad g = \omega\,, \qquad h = \Omega \tag{19}$$

where $\mu = f\,(M + E)$, f is the gravitational constant, E the mass of the earth, M the mass of the moon. The equations of motion are:

$$\frac{dL}{dt} = +\frac{\partial R}{\partial l}\,, \quad \frac{dG}{dt} = +\frac{\partial R}{\partial g}\,, \quad \frac{dH}{dt} = +\frac{\partial R}{\partial h} \tag{20}$$

$$\frac{dl}{dt} = -\frac{\partial R}{\partial L}\,, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}\,, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H}\,, \tag{21}$$

$$R = \frac{\mu^2}{2L} + \Omega\,,$$

Ω being the original disturbing function and it is developable into a trigonometric series in arguments l, l', g, and h with the coefficients depending upon L, G, and H. By a chain of properly chosen canonical transformations, Delaunay removes the short periodic terms from R, one by one, starting with the terms of the lower order until all the significant short periodic terms are removed and only the secular term is left. In going from one canonical transformation to the other, Delaunay retains for the canonical variables the same notations L, G, H, l, g, h, throughout. However, the meaning of these notations is moving away, more and more, after each canonical transformation, from their original meaning as given by the Eq. (18) and (19). After short period the arguments are eliminated from R, ~~and~~ the Eq. (20) become

$$\frac{dL}{dt} = 0, \qquad \frac{dG}{dt} = 0, \qquad \frac{dH}{dt} = 0$$

and, consequently, the final L, G and H become the constants of integration. They serve to define the mean values of the elements e, i and a. It follows from Eq. (21) that the final l, g, and h are the linear functions of time

$$\begin{aligned} l &= -\frac{\partial R}{\partial L} t + l_0 \\ g &= -\frac{\partial R}{\partial G} t + g_0 \\ h &= -\frac{\partial R}{\partial H} t + h_0 \end{aligned} \tag{22}$$

where l_0, g_0 and h_0 are the constants of integration defining the original phases.

The final outcome of the theory of Delaunay are the trigonometrical series for the true longitude V, the latitude , and the paralax in terms of the four arguments of the forms:

$D = (n-n')t + \varepsilon - \varepsilon'$ = half of the arg. of the variation,

$l = cnt + \varepsilon - \varpi$ = the arg. of the principal elliptical term

$l' = n't + \varepsilon' - \varpi'$ = arg. of the "annual inequality"

$F = gnt + \varepsilon - \theta$ = mean arg. of the latitude,

ε is the mean longitude of the moon, ϖ is the mean longitude of the perigee, and θ the mean longitude of the node at t = 0 and

$$(1-c)nt + \varpi,$$

$$(1-g)nt + \theta$$

are the mean position of the perigee and of the node, ε' is the mean longitude of the sun and ϖ' – the longitude of the perigee at t = 0. The series for V, U, and a/r have the form

$$U = \sum B e^p e'^{p'} \gamma^q \alpha^r m^s \sin(iD + jl + j'l' + kF),$$

$$V = nt + \varepsilon_0 + \sum A e^p e'^{p'} \gamma^q \alpha^r m^s \sin(iD + jl + j'l' + kF)$$

$$\frac{a}{r} = \sum C e^p e'^p \gamma^q \alpha^r m^s \cos(iD + jl + j'l' + kF),$$

where A, B, and C are the numerical coefficients; p, p', q, r, and s are non-negative integers; i, j, j', and k are integers which can take values from $-\infty$ to $+\infty$; e is the mean eccentricity of the lunar orbit; $\gamma = \sin\frac{i}{2}$, where i is the mean inclination of the lunar orbit plane toward the orbit plane of the sun; $\alpha = a/a'$ is the parallactic factor and $m = n'/n$ where n and n' are the observed mean sidereal motions of the moon and of the sun, respectively. The actual development was carried out up to the seventh order with respect to the small parameters. In the case of the moon, the series for V, the longitude, and U, the latitude, contains more than 400 terms and the series for a/r contains about 100 terms.

Delaunay performed more than 600 canonical transformations until he achieved the elimination of all significant short periodic terms. It took him about twenty years to accomplish this work. One can only admire his persistence and courage in undertaking such a formidable task and in bringing it to a successful completion. The series for V, U, and a/r represent the most complete algebraic solution of the satellite problem ever acheived.

The theory of Delaunay was applied to the VI, VII, and X satellites of Jupiter and recently to the investigations of motion of the hypothetical satellites of the moon as disturbed by the earth.

Delaunay lived long enough to complete only the development of the direct solar effects. The secondary effects, the effect of the deviation of the solar motion from the elliptic one, the planetary effects, etc., are not included in his theory. If someone in our day will decide to repeat and to extend the work of Delaunay, he will choose the methods of Poincaré, von Zeipel or Brown. These methods permit one to remove from the disturbing functions the periodic terms in groups, instead of just one term, as it is done in the classical version of the theory of Delaunay. The recent results by Hori based on the application of von Zeipel's method, permit one to conclude that the computational process in lunar theory can by sped up considerably.

Also, there is the tendency now not to perform the development in terms of the mean anomaly and in powers of the eccentricity, but to keep the development in a more closed form by writing it in terms of the true or of the eccentric anomalies.

The work by Brown (1937) on the stellar three-body problems indicates the possibility of extension of Delaunay's solution to the case of a large eccentricity and a large inclination of the orbital plane of the satellites.

The theory of Hansen (Theoria motus, 1838, and Darlegung, 1857, 1858) requires the analytical expansion of the disturbing function, but otherwise the process of developing the perturbations is a purely numerical one.

The output are the trigonometric series in four arguments with the purely numerical coefficients.

The method of Hansen does not require that the first approximation to the solar ~~motion~~ trajectory be an ellipse. It permits an easy inclusion of the perturbations from every source, and finally, it permits one to treat the high orbital inclinations.

The theory is very adaptable to the use of automatic electronic computers. It can be used to develop the theories of satellites of outer planets, and one might hope that some day the Hansen lunar theory will be checked and some of the coefficients will be corrected using an electronic computer.

Hansen splits the perturbations of the satellite into the perturbations of the osculating orbital plane and into the perturbations of the satellite in that plane. These two types of perturbations are not completely independent of each other, but their mutual effects are of a higher order and they are taken into consideration. The perturbations in the osculating orbital plane are treated by referring the motions of the satellite to a rotating system of coordinates rigidly connected with that plane.

The osculating orbital plane rotates together with this system, like a rigid body, around the instantenous positions vector of the satellites. The system of coordinates rigidly connected with the osculating orbit plane is termed as ideal by Hansen, because the vectors of the relative and of the absolute velocity coincide and also because the equations of the relative motions have the same form as in the inertial system. We put the $\underline{x}$ and $\underline{y}$ axes into the osculating orbit plane $\underline{X}$. The intersection of the x-axis with the celestial sphere, is called the "departure point" by Hansen. All angles in the orbit plane, like the true orbital longitude of the satellite v and the true orbital longitude of the perigee X are reckoned from the departure point.

A similar system of coordinates with its departure point X' can also be defined in the instanteneous orbital plane of the sun.

As an intermediary solution, Hansen introduces in the osculating orbital plane an auxilary satellite, moving in accordance with Kepler's laws in an ellipse of the constant shape. This ellipse is caused to rotate with the constant angular velocity n_0t relative to the ideal system of coordinates.

The elements of the ellipse and its angular velocity of rotation are chosen in such a way that no secular or mixed terms appear in the development of the coordinates. The position vector of the real satellite is then determined by its deflection in time and space from the position of the auxiliary satellite. Let $\vec{r}$ (t) be the position vector of the real satellite at the moment t. There exists a moment z such that the position vector $\vec{\bar{r}}(z)$ of the auxiliary satellite has the some direction as $\vec{r}(t)$. In other words we have the relation

$$\vec{r}\,(t) = (1+\nu)\,\vec{\bar{r}}\,(z)$$

where $(1 + \nu)$ is a scalar factor. The quantities $\nu\bar{r}$ and δz are interpreted as the perturbations of the radius vector and of time. The perturbations of time are included in the mean anomaly of the auxiliary satellite and Kepler's equation for this satellite can be written in the form

$$\varepsilon - e_0 \sin \varepsilon = c_0 + n_0 t + n_0 \delta z = c_0 + n_0 \bar{z}$$

where n_0 is the anomalistic mean motion of the real satellite. The angle $n_0 \bar{z}$ contains all the purely periodic effects in the osculating mean anomaly and in the true orbital longitude . Thus, all the purely periodic angular perturbations of the real satellite in its instanteneous orbit plane are combined into one angle: the perturbations of the mean anomaly of the auxiliary satellite.

Let h^{-1} be the areal velocity of the real satellite, a_0 – the semiaxes major, e_0–the eccentricity of the auxiliary ellipse, h_0^{-1}–the areal velocity of the auxiliary satellite, $\bar{f}$ its true anomaly. It is of great interest and importance that in Hansen's lunar theory the perturbations in the orbital plane, v and $n_0 \delta z$, can be expressed in terms of one single functions W defined by the equation

$$\overline{W} = -1 - \frac{h_0}{h} + 2\frac{h}{h_0} \cdot \frac{\bar{r}}{a_0} \frac{1 + e\cos(\bar{f} + n_0 y t - \chi)}{1 - e_0^2}$$

It can be shown that

$$\frac{d\delta z}{dt} = \dot{\overline{W}} + \frac{h_0}{h}\left(\frac{v}{1+v}\right)^2 - \frac{y}{\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2$$

and

$$\frac{d\nu}{dt} = -\frac{1}{2}\frac{\partial \bar{W}}{\partial z} + \frac{1}{2}\frac{y(1+\nu)}{\sqrt{1-e^2}}\frac{d}{dz}\left(\frac{r}{a_o}\right)^2$$

In the process of formation of the differential equation for W it is convenient to consider $\bar{r}$ and $\bar{f}$ as temporary constants. This is done by replacing $\bar{r}$ and $\bar{f}$ by the symbols $\bar{\rho}$ and $\bar{\varphi}$, considered as the functions of the temporarily "frozen" mean anomaly $c_o + n_o t + n_o \delta\zeta$. Correspondingly, the notation $\bar{W}$ is modified to W. Considered as a function of the time t the W-function depends now only upon the osculating elements. Thus, the differential equation for W can be formed using the theory of variations of astronomical constants. After the integration Hansen's "bar operations" is performed on W-function. This operation consists in replacing in W, τ by t, $\delta\zeta$ by δz, and, consequently, $\bar{\rho}$ and $\bar{\varphi}$ by $\bar{r}$ and $\bar{f}$, W by $\bar{W}$ again.

Let Ψ and Ψ' be the angular distances of the departure points X and X' from the common node of the lunar and of the solar orbit planes. These angles are affected by the secular and by the purely periodic perturbations, and we can set

$$\Psi = \Pi_o - n_o(\alpha - \eta)\,t - (K + N)$$
$$\Psi' = \Pi_o' - n_o(\alpha + \eta)\,t + (K - N)$$

where Π_o, Π_o', α and η are constants and K and N are purely periodic. The elements of the satellite and the secular motions of Ψ and Ψ' must be determined in such a way that no secular or mixed terms are present in the development of the coordinates. The mean motions -

$-n_0(\alpha-\eta)$ and $-n_0(\alpha+\eta)$ are constants as long as only the direct solar action is considered. The planetary effects produce small accelerations in the mean motions of the node and of the perigee and terms containing $t^2, t^3, \ldots$ appear. These are taken into account by writing $\int n_0 \gamma\, dt$, $\int n_0 \alpha\, dt$, $\int n_0 \eta\, dt$ instead of $n_0\gamma, n_0\alpha, n_0\eta$. Let J be the angle between the lunar and the solar orbit planes. Hansen makes use of the parameters

$$P = 2 \sin \frac{J}{2} \sin \mathcal{N}, \quad Q = 2 \sin \frac{J}{2} \cos \mathcal{N}, \text{ and } K$$

which carry all the periodic effects in the position of the lunar orbit plane. The mean motion $n_0 \alpha$ and $n_0 \eta$ are determined from the differential equations for these parameters in such a way that neither P nor K contain any secular effects.

Instead of P, Q and K the set of "redundant" parameters

$$\lambda_1 = \sin \frac{J}{2} \cos \mathcal{N}, \qquad \lambda_3 = \cos \frac{J}{2} \sin K$$

$$\lambda_2 = \sin \frac{J}{2} \sin \mathcal{N} \qquad \lambda_4 = \cos \frac{J}{2} \cos K$$

satisfying the condition

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$$

can be used. Their introduction makes the development more symmetric and algebraic in form and leaves in the arguments only the linear functions of time from the very start of the computational procedure.

The method of solution used by Hansen is one of sucessive approximations. The right sides of the equations for dW/dt, dP/dt, dQ/dt, dK/dt, are developed in powers of ν, $n_0\delta z$ and in powers of the perturbations δP, δQ, δK

The first approximation to these quantities is obtained by integrating the simplified differential equations, neglecting the perturbations in the expressions for the derivatives. Then the integration is repeated with the better values of the derivatives, until the final numerical results are reached. The final results are the trigonometric series for $n_0 \delta z$, log $(1 + \nu)$ and the latitude of the moon in terms of the arguments:

$$\begin{aligned} g &= n_0 t + g_0 , \\ g' &= n_0' t + g_0' , \\ \omega &= n_0 (y + \alpha - \eta) t + \omega_0 , \\ \omega' &= n_0 (y' + \alpha + \eta) t + \omega_0' , \end{aligned}$$

where g is the mean anomaly of the Moon, g' the mean anomaly of the sun, ω the mean argument of the perigee and $-\omega'$ is the mean node of the lunar orbit relative to the departure point in the solar orbit plane.

In the past, Hansen's lunar theory was considered as one difficult to understand, partly because of the way it was presented and partly because of its unusual way of treating the perturbations. It seems, that the idea of the "disturbed time" was especially repugnant. From our present standpoint, Hansen's theory is mathematically very clear. At several institutes work is being done on application of the machines to develop Hansen's theory of the "lost" satellites in the solar system and to obtain their corrected elements and prediction of their positions in the sky.

The modern lunar theory by Hill and Brown gives the expansion of the rectangular coordinates of the moon in a rotating geocentric system. The main problem of determining the direct solar effects is solved first, then the corrections for several secondary effects are added. In the main problem we take the motion of the sun to be elliptic; consequently, the plane of the ecliptic is considered as fixed.

The x-axis of the rotating system is directed toward the mean sun, the x and y-axes are placed into the ecliptical plane. The angular velocity of rotation of the system is equal to observed mean motion of the sun.

Let x, y, z be the coordinates of the moon, r its distance from the earth, ρ the projection of r on the xy -plane; M the mass of the moon, E the mass of the earth and m' the mass of the sun; and n the observed mean motion of the moon, n' of the sun; e and k the constants of the eccentricity and of the inclinations which appear in the expansion of the lunar coordinates; r', e', a' the radius vector, the eccentricity and the semimajor axis of the solar orbit, respectively.

If one neglects in the expansion of the lunar coordinates, e.g., in the expansion by Delauney, the terms depending upon the constants of integration e, k, α and e' then the remaining terms depend only upon the half of the "argument of the variation"

$$D = (n - n')(t - t_0) + \varepsilon - \varepsilon'.$$

This shows that the group of inequalities depending only upon $\underline{D}$ can serve as a basic intermediary solution being independent from the constants of integration e, k, α and e'. This is called the "variational solution" because it contains only the argument of the variation. It can be found by solving the set of differential equations which determine this intermediary solution.

In Hill's theory $m = n'/(n - n\ )$ is present as a small parameter in the differential equation of motion. The convergence in powers of this parameter is faster in the parameter of Delauney n'/n. The basic arguments become

$$
\begin{aligned}
D &= (n - n')(t - t_0), \\
l &= c(n - n')(t - t_1), \\
l' &= m(n - n')(t - t_3), \\
F &= g(n - n')(t - t_2), \\
c &= m\underline{c}, \quad g = m\underline{g}
\end{aligned}
$$

where c and g are the series in e^2, k^2, α^2, e'^2. The initial moments of time t_0, t_I, t_2, t_3 can be omitted from the development without any harm. They are re-introduced after the problem is solved and series expansions for the coordinates are found.

It is easier to operate with the power series than with the trigonometric series. For this reason Hill introduces the exponential function

$$\zeta = \exp i(n - n')(t - t_0)$$

From the form of the arguments it is evident that the coordinates of the moon can be represented as the power series in

$$\zeta, \zeta^c, \zeta^g, \zeta^m$$

and also as the power series in e, k, α, e'. Instead of x and y, Hill is introducing the complex coordinates

$$u = x + iy, \quad s = x - iy, \quad i = -1$$

then the disturbing function Ω_1 of the Hill-Brown lunar theory is developable into a power series in u, s, z, and ζ^m. Ω_1 becomes zero if e' and α are neglected. Introducing the differential operator

$$D = \zeta \frac{d}{d\zeta}$$

the classical differential equation of motion in a rotating system can be reduced to the form

$$\begin{aligned}
(D+m)^2 u + \frac{1}{2} m^2 u + \frac{3}{2} m^2 s - \kappa u/(us+z^2)^{3/2} &= -\frac{\partial \Omega_1}{\partial s} \\
(D-m)^2 s + \frac{1}{2} m^2 s + \frac{3}{2} m^2 u - \kappa s/(us+z^2)^{3/2} &= -\frac{\partial \Omega_1}{\partial u} \qquad (23) \\
(D^2-m^2) z - \kappa z/(us+z^2)^{3/2} &= -\frac{1}{2} \frac{\partial \Omega_1}{\partial z}
\end{aligned}$$

which is very convenient to obtain the solutions in the form of the power series.

The series representing the coordinates of the moon have the form

$$u\zeta^{-1},\ s\zeta^{+1},\ z\,i = \sum A(m)\, e^{|p|+2p'}\, e'^{\,|r|+2r'}\, k^{|q|+2q'}\, \alpha^{s'} \times \zeta^{2i+pc+rm+qg}$$

$$2i,\ p,\ q,\ r = 0,\ \pm 1,\ \pm 2,\ \ldots\ldots,\qquad p',\ r',\ q',\ s' = 0,\ 1,\ 2,\ \ldots\ldots,$$

where A is a function of m only, $2i$ and s' are of the same parity, q is even in $u\zeta^{-1}$, $s_\tau\zeta^{+1}$ and it is odd in $z\,i$.

The factor

$$e^{|p|+2p'}\, e'^{\,|r|+2r'}\, k^{|q|+2q'}\, \alpha^{s'}$$

is called the characteristic of the term in the expansion and

$$|p| + 2p' + |r| + 2r' + |q| + 2q' + s'$$

is called the order. Arranging the series in orders and characteristics we can write

$$u = u_0 + (e\,u_e + e'\,u_{e'} + \alpha\,u_\alpha) + (e^2 u_{e^2} + e e' u_{ee'} + e'^2 u_{e'^2} + e\alpha\, u_{e\alpha} + e'\alpha\, u_{e'\alpha} + \alpha^2 u_{\alpha^2} + k^2 u_{k^2}) + \cdots\cdots,$$

$$s = s_0 + (e\,s_e + e'\,s_{e'} + \alpha\,s_\alpha) + (e^2 s_{e^2} + e e' s_{ee'} + e'^2 s_{e'^2} + e\alpha\, s_{e\alpha} + e'\alpha\, s_{e'\alpha} + \alpha^2 s_{\alpha^2} + k^2 s_{k^2}) + \cdots\cdots,$$

$$z = z_k + (z_{ke} + z_{ke'} + z_{k\alpha}) + \cdots\cdots$$

Substituting these series into the equation of motion we deduce

$$(D+m)^2 u_o + \frac{1}{2} m^2 u_o + \frac{3}{2} m^2 s_o - \frac{\kappa u_o}{\rho_o^3} = 0$$
$$(D-m)^2 s_o + \frac{1}{2} m^2 s_o + \frac{3}{2} m^2 u_o - \frac{\kappa s_o}{\rho_o^3} = 0 \qquad (23')$$
$$\rho_o^2 = u_o s_o$$

and the set of the differential equations of the form

$$\zeta^{-1}(D+m)^2 u_\tau + M(u_\tau \zeta^{-1}) + (N\zeta^{-2})(s_\tau \zeta^{+1}) = A_\tau, \qquad (24)$$
$$D^2 z_\tau - 2M z_\tau = B_\tau, \qquad (25)$$

where

$$M = \frac{1}{2} m^2 + \frac{1}{2} \frac{\kappa}{\rho_o^3} = \sum_i M_i \zeta^{2i}, \qquad (26)$$

$$N = \frac{3}{2} m^2 + \frac{3}{2} \frac{\kappa u_o^2}{\rho_o^5} = \sum_i N_i \zeta^{2i} \qquad (27)$$

The unknowns u_τ, s_τ, z_τ and also $\underline{A}_\tau$, $\underline{B}_\tau$, have the characteristic τ.

$\underline{A}$ and $\underline{B}$ have the form

$$A = \underline{\underline{a}} \sum_i (A_i \zeta^{2i+\tau} + A_i' \zeta^{2i-\tau})$$

$$B = \underline{\underline{a}} \sum_i B_i (\zeta^{2i+\tau} - \zeta^{-2i-\tau})$$

where $2i$ is either 0, ± 2, ± 4, ...or ± 1, ± 3, ... They are obtained as a result of an expansion of Eq. (24) and (25) and as a result of the algebraical operations performed on the terms having the characteristics lower than τ.

The terms with the characteristics zero u_0, s_0 are to be determined first. Then we resort to the Eqs. (24) and (25) for the successive determination of the terms of orders 1, 2, 3, ... and of all admissible characteristics inside of each order.

The solution of the zero order must have the form

$$u_0 = \underline{a} \sum_i a_{2i} \zeta^{2i+1}, \qquad s_0 = \underline{a} \sum_i a_{-2i-2} \zeta^{2i+1}.$$

Many methods were proposed for the numerical computation of the coefficients. It seems that the original method by Hill, based on the solution of an infinite set of the linear equations with the coefficients rational in m, still remains a most convenient method from the computational standpoint. The solution is greatly facilitated by the fact that

$$a_{2i} = O(m^{2|i|})$$

and a_{2i} decreases very fast as i increases.

For the moon, Hill obtained u_0 and s_0 accurate up to 10^{-15}, and he found that only thirteen terms are to be kept.

The terms of the first order with the characteristics e and k are especially important in the determination of the main parts of the motion of the lunar perigee and node. These terms satisfy the differential equations

$$\zeta^{-1}(D+m)^2 u_e + M(u_e \zeta^{-1}) + (N\zeta^{-2})(s_e \zeta^{+1}) = 0$$

and

$$D^2 z_k - 2 M z_k = 0$$

Evidently, $e u_e$ and $e s_e$ can be considered as the variations δu_0 and δs_0 of the coordinates u_0 and s_0, and the equation written above can be obtained by the application of the $\delta-$ operator to Eq. (23'). The variations of the coordinates can be decomposed along the tangent and along the normal to the curve representing the solution of the zero order. Because of the periodicity and the symmetry of this solution, one has to expect that the variation along its normal will satisfy the equation of the form

$$D^2 \delta \mathcal{N} = \Theta \delta \mathcal{N}$$

where

$$\Theta = \sum_i \Theta_i \zeta^{2i}$$

and

$$\Theta_i = \Theta_{-i}$$

This is the celebrated Hill's equation.

A lengthy computation shows that

$$\Theta = -\left(\frac{K}{\rho^3} + m^2\right) + 2\left\{\frac{1}{2}\left(\frac{D^2 u_0}{D u_0} - \frac{D^2 s_0}{D s_0}\right) + m\right\}^2 - \left\{\frac{1}{2}\left(\frac{D^2 u_0}{D u_0} + \frac{D^2 s_0}{D s_0}\right)\right\}^2 - D\left\{\frac{1}{2}\left(\frac{D^2 u_0}{D u_0} + \frac{D^2 s_0}{D s_0}\right)\right\}.$$

The general solution of Hill's equation is

$$\delta \mathcal{N} = c_{+1} \sum_i b_i \zeta^{2i+c} + c_{-1} \sum_i b_i' \zeta^{2i-c}$$

where c_1 and c_{-1} are two arbitrary constants and b_i, b_i' are functions of $\underline{m}$; c is the root of the determinatal equation

$$\Delta(c) = \det \left\{ \frac{(c+2i)^2 \delta_{ij} - \Theta_{j-i}}{4j^2 - \Theta_0} \right\} \tag{30}$$

where

$$i, j = \ldots, -2, -1, 0, +1, +2, \ldots$$

and δ_{ij} are Kronecker's deltas. This equation can be brought to the form

$$\sin^2 \tfrac{1}{2}\pi c = \Delta(0) \sin^2 \tfrac{1}{2}\pi\sqrt{\Theta_0}$$

Hill obtained the value

$$c_0 = 1.07158\ 32774\ 16012$$

for $\underline{c}$ accurate to fifteen places. Of course, this value, in fact, represents only the main part of $\underline{c}$. The higher order terms, depending upon $\underline{e}^2$, $\underline{k}^2$, $\underline{e}'^2$ and $\underline{u}^2$, ..., will be added to $\underline{c}_0$ in the process of computation of terms with higher characteristics. The determinants $\Delta(\underline{c})$ and $\Delta(0)$ have the infinite number of rows and columns. A concise, but a descriptive, account on convergence of infinite determinants can be found in Whittaker and Watson "Modern Analysis".

The introduction of infinite determinants into analysis by Hill was a daring step, which helped him to solve a difficult problem in the lunar theory. However, in the practical computations, say for the satellites of Jupiter, the determination of $\underline{c}_0$ using the process of successive approximations would be preferable. The fast convergent procedures were developed by Andoyer and Brown. The forms of $\underline{N}$, and M suggest that the substitution

$$u_e \zeta^{-1} = a \sum_i \left(\varepsilon_i \zeta^{2i+c} + \varepsilon_i' \zeta^{2i-c} \right)$$

$$i = 0, \pm 1, \pm 2, \ldots\ldots$$

into Eq. (28) can be tried after the main part of $\underline{c}$ is found. We obtain for ε_i, ε_i' an infinite set of the homogeneous linear equations. They must be compatible and the condition for their compatibility is already given by Eq. (30). This system must be amended by one more equation. In the Hill-Brown lunar theory this additional equation is

$$\varepsilon_0 - \varepsilon_0' = 1$$

As a consequence of this additional condition, the constant of the eccentricity of the Hill-Brown lunar theory is almost twice the constant of the Delaunay theory.

We rearrange the system of equations for ε_i, ε_i' into a set of pairs. For each pair of unknowns ε_i, ε_{-i}' we can find a pair of equations with the coefficients of ε_i, ε_{-i}' dominant over the coefficients of the remaining unknowns. The system arranged this way can be solved by the method of successive iteration.

A similar procedure is used for the computation of terms of a higher characteristic τ, providing that terms of the lower characteristics have been determined. We make the substitutions

$$u_\tau \zeta^{-1} = \underline{\underline{a}} \sum_i (\lambda_i \zeta^{2i+\tau} + \lambda_i' \zeta^{2i-\tau})$$

$$z_\tau = \underline{\underline{a}} \sum_i \mu_i (\zeta^{2i+\tau} - \zeta^{-2i-\tau})$$

and arrange the linear equations for λ_i, λ'_i in pairs, similary as for the terms of the characteristic $\underline{e}$. The

$$[(2i+1+\tau+m)^2 + M_0]\lambda_i + N_0 \lambda'_{-i},$$

$$N_0 \lambda_i + [(2\tau-1+\tau-m)^2 + M_0]\lambda'_{-i}.$$

When we solve the system of equations for λ_i, λ_i by the method of iteration, we find a difficulty, caused by the presence of a small divisor, will arise when the expression

$$[(2i+\tau+1+m)^2 + M_0][(2i+\tau-1-m)^2 + M_0] - N_0^2$$

becomes small. Similarly, we have the difficulty in determining μ_i when

$$(2i+\tau)^2 - 2M_0$$

becomes small. We have many such terms, of the short and of the long periods, in all characteristics. One of them is the evection in the group of the characteristic $\underline{e}$, it has the form

$$\varepsilon_{-1}\zeta^{+2-c} + \varepsilon'_{+1}\zeta^{-2+c}.$$

The most troublesome term in u and s is the term ζ^{2g-2c}, having the Delaunay argument $2F - 2\ell$. This is a most difficult term in Hansen, as well as in Hill-Brown lunar theory. The test which any lunar theory, numerical or analytical, must stand against, is the accuracy in producing the long-period effects. All authors, Hansen, Hill and Brown developed special precedures for correcting the first approximations to the coefficients of the critical terms. In the Hill-Brown theory the homogeneous form of the equations of motion is used for this purpose. The main portion g_0 of the secular motion of the node is obtained from the differential equation for z_k. Adams has developed a method for the determination of g_0, which is akin to Hill's method for c_0. The complete values of c and g are the multiple power series in e^2, k^2, e'^2 and α^2. Thus writing only the first few terms,

$$c = c_0 + e^2 c_{e^2} + k^2 c_{k^2} + e'^2 c_{e'^2} + \alpha^2 c_{\alpha^2} + \cdots$$

$$g = g_0 + e^2 g_{e^2} + k^2 g_{k^2} + e'^2 g_{e'^2} + \alpha^2 g_{\alpha^2} + \cdots$$

The determination of, say, c_{e^2} is based on the fact that the expression for u_{e^3} contains the arguments which are appearing also in u_e. The terms with the characteristic e and having the factor c_{e^2} will appear in the differential equation for u_{e^3} as the consequence of the application of the operators D and D^2 to u_e. These terms (with the factors e^3 omitted) must be shifted from the differential equation for u_e to the differential equation for u_{e^3}. Then, similarly as before, we obtain a set of the linear eequations of the coefficients at $\zeta^{2i\pm c}$ (i = 0, 1, 2, ...) in u_{e^3}. The compatibility of these equations leads to an equation linear in c_{e^2}. Thus c_{e^2} can be determined. Similar considerations are valid for

$$c_{e'^2}, \quad c_{k^2}, \quad c_{\alpha^2}, \ldots$$

As the final result, Brown's tables give the solar parts of the true longitude, the latitude and of the parallax in terms of the arguments of Delaunay's l, l', F and D.

After the principal part of the lunar theory, the solar perturbations, were developed, the effects of the shapes of the earth and of the moon, the planetary effects and the effect of the deviation of the motion of the sun from the elliptic motion were also included. Despite all the enormous work done by Brown, the discrepancies between the purely gravitational theory and the observations still remained. In order to remove them, Brown had to introduce a purely empirical periodic term into the development of the lunar longitude. Already, Halley, on the basis of the ancient eclipses, has found the existence of the secular acceleration in the mean motion of the moon. Laplace attributed it to the variability of the eccentricity of the earth's orbit, which is caused by the planetary perturbations. This value for the acceleration was confirmed by Plana, Damoiseau and Hansen, but Adams found that the purely gravitational acceleration is only nearly a half of the value obtained by Laplace.

The investigations by Fotheringham, de Sitter and Spenser-Jones point out the irregularities in the rotation of the earth as a cause of the discrepancy between observed and the computed positions of the moon. The tidal friction, especially in the enclosed and narrow seas, and the variability of the moments of inertia of the earth cause the difference between the dynamical time, in the equations of motion, and the mean solar time to change.

The gravitational theory of the moon is referred to the dynamical time, but the observations are referred to the mean solar time. In order to account for this difference and to bring the ephemeris of the moon into agreement with the ephemeris of the sun, the empirical term is removed from the mean longitude of the moon as it is given in Brown's table and the correction

$$-8''.72 - 26''.74T - 11''.22T^2$$

is introduced instead. The difference between the dynamical time and the mean solar time is

$$T = +24^s.349 + 72^s.318T + 29^s.950T^2 + 1.82144...B$$

This correction for the irregularities of the earth's rotation includes the effect of the tidal friction as well as the effect of the variability of the moments of inertia of the earth. This last effect is carried by B, which is called "the fluctuation of the moon in the longitude" and represents the effect of the variability of moments of inertia on the longitude of the moon. A complete discussion of the subject can be found in the book by Munk and MacDonald (1960). Recently W. Eckert from Watson Computing Laboratory undertook the revision of the Hill-Brown lunar theory. He corrected the numerical values of the coefficients as well as the values of the original elements. With these important corrections of the main problem, we have a highly accurate lunar theory which can serve as a solid foundation for the determination of the secondary effects as of the planetary effects and of the effects caused by the fugures of the earth and the moon. The direct planetary periodic perturbation in the longitude is produced by Venus. It has about 17" in the amplitude. The upper limit for the coefficients in the perturbations caused by the figure of the earth is about 9". Most significant are the terms having as the arguments the longitude of the node and the mean longitude of the moon.

Artificial Satellites.

The theory of artificial satellites constitutes now a special chapter in celestial mechanics. We shall consider here the purely gravitational effects which can be treated using the classical means. The main perturbative effect in the motion of artificial satellites is caused by the presence of the equatorial bulge. If this bulge were not present and the earth were a sphere composed of homogeneous layers, then the motion of a satellite would be purely elliptic. The force function would be of the form

$$U = \frac{\mu}{r}$$

where r is the distance of the satellite from the center of the earth. If the bulge is present, then the force function becomes

$$U = \mu \int \frac{dm}{|\vec{\rho} - \vec{r}|}$$

where $\vec{\rho}$ is the position vector of the particle of the earth from its center of mass G and $\vec{r}$ is the position vector of the satellite relative to G. The integral is taken over the volume of the earth. If we assume the axial symmetry of the earth, then U can be developed into a series of Legendre polynomials (zonal harmonics) and we have

$$U = \mu \left[\frac{1}{r} + \frac{J_2}{r^3} P_2 \left(\frac{z}{r} \right) + \frac{J_3}{r^4} P_3 \left(\frac{z}{r} \right) + \frac{J_4}{r^5} P_4 \left(\frac{z}{r} \right) + \dots \right], \quad (31)$$

where z is the distance of the satellite from the equatorial plane. Because of the perturbative action of the bulge, the orbit is not a fixed ellipse and the elements become the functions of time. Of all the perturbative effects represented in Eq. (31), the dipole term

$$\frac{\mu J_2}{r^3} \underline{P}_2 \left(\frac{z}{r}\right)$$

is the most significant one. The problem given by the simplified disturbing function

$$U = \frac{\mu}{r} + \frac{\mu J_2}{r^3} \underline{P}_2 \left(\frac{z}{r}\right) \tag{32}$$

constitutes the so-called main problem in the theory of artificial satellites. The remaining perturbative effects constitute small contributions to the main problem and at the present time can be treated differentially. The most elegant purely analytical solution of the artificial satellite problem is given by Brower (1959) and his school. The problem is solved using von-Zeipel's method of elimination of short-period terms.

In a previous section, we mentioned the canonical elements of Delaunay: L, G, H, l, g, h. Brouwer makes use of them and writes the equations for variation of elements in the form

$$\frac{dL}{dt} = + \frac{\partial F}{\partial l}, \quad \frac{dl}{dt} = - \frac{\partial F}{\partial L}$$

$$\frac{dG}{dt} = + \frac{\partial F}{\partial g}, \qquad \frac{dg}{dt} = - \frac{\partial F}{\partial G}$$

$$\frac{dH}{dt} = + \frac{\partial F}{\partial h}, \qquad \frac{dh}{dt} = - \frac{\partial F}{\partial H}$$

where the Hamiltonian F is in Brouwer's notations

$$F = \frac{\mu^2}{2L^2} + \frac{\mu^4 k_2}{L^6} \left[\left(- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G^2} \right) \frac{a^3}{r^3} + \right.$$
$$\left. + \left(\frac{3}{2} - \frac{3}{2} \frac{H^2}{G^2} \right) \frac{a^3}{r^3} \cos(2g + 2f) \right]$$

where f is the true anomaly of the satellite.

In fact F is the force function [Eq. (32)] expressed in terms of the elements of Delaunay. By applying a properly chosen canonical transformation

$$(L, G, H, l, g, h) \rightarrow (L', G', H', l', g', h')$$

$$L = \frac{\partial S}{\partial l}, \qquad G = \frac{\partial S}{\partial g}, \qquad H = \frac{\partial S}{\partial h}$$

$$l' = \frac{\partial S}{\partial L}, \qquad g' = \frac{\partial S}{\partial G}, \qquad h' = \frac{\partial S}{\partial H}$$

$$S = L'l + G'g + Hh + S_1 (L', G', H, l, g) +$$
$$+ S_2 (L', G', H, l, g) + O(k_2^3)$$

Brouwer eliminates all the short-period term from the Hamiltonian, thus reducing it to the new form

$$F^* = \frac{\mu^2}{2L'^2} + \frac{\mu^4 k_2}{L'^3 G'^3} \left(- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G'^2} \right)$$

$$+ \frac{\mu^6 k_2^2}{L'^{10}} \left[+ \frac{15}{32} \frac{L'^5}{G'^5} \left(1 - \frac{18}{5} \frac{H^2}{G'^2} + \frac{H^4}{G'^4} \right) + \right.$$

$$+ \frac{3}{8} \frac{L'^6}{G'^6} \left(1 - 6 \frac{H^2}{G'^2} + 9 \frac{H^4}{G'^4} \right) -$$

$$\left. - \frac{15}{32} \frac{L'^7}{G'^7} \left(1 - 2 \frac{H^2}{G'^2} - 7 \frac{H^4}{G'^4} \right) \right] +$$

$$+ \frac{\mu^6 k_2^2}{L'^{10}} \left[- \frac{3}{16} \left(\frac{L'^5}{G'^5} - \frac{L'^7}{G'^7} \right) \cdot \right.$$

$$\left. \cdot \left(1 - 16 \frac{H^2}{G'^2} + 15 \frac{H^2}{G'^4} \right) \right] \cos 2g'$$

An additional canonical transformation removes the long-period terms and we arrive at a Hamiltonian of the form

$$F^{**} = F^{**}(L'', G'', H)$$

which does not contain the periodic terms at all. With the accuracy up to k_2^2, the Hamiltonian F^{**} has exactly the same form as the purely secular part in F^*. Only L', G' are replaced by L'' and G''.

From the new canonical equations

$$\frac{dL''}{dt} = + \frac{\partial F^{**}}{\partial l''} = 0, \qquad \frac{dl''}{dt} = - \frac{\partial F^{**}}{\partial L''}$$

$$\frac{dG''}{dt} = + \frac{\partial F^{**}}{\partial g''} = 0, \qquad \frac{dg''}{dt} = - \frac{\partial F^{**}}{\partial G''}$$

$$\frac{dH}{dt} = + \frac{\partial F^{**}}{\partial h''} = 0, \qquad \frac{dh''}{dt} = - \frac{\partial F^{**}}{\partial H}$$

it is evident that L'', G'' and H are constants and that the elements l'', g'' and h'' are linear functions of time. These elements are the mean elements of Brouwer's theory. The original disturbing function F does not contain the long-period term. However, this term appears in the transformed Hamiltonian F^*, but with a very small factor k_2^2. Thus, the order of magnitude of the long-period term in F^* is higher than the order of the corresponding purely secular term. This is exactly the reason why the artificial satellite problem is solvable in terms of Fourier series. In the lunar problem both terms can be of the same order, and consequently, solution in the form of a trigonometric series will be not always possible.

The additional corrections as caused by the presence of the third, fourth and fifth harmonics in the external potential of the earth are added to the main solution in a differential way. All these harmonics will introduce some additional long-period terms in the elements, but only the fourth harmonic will contribute to the mean motions of the basic arguments l'', g'' and h''. Final expressions of these mean motions are given with the accuracy up to k_2^2. Thus, the solution for the disturbed elements is obtainable in the form of asymptotic series in the small parameter k_2. The periodic terms proportional to k_2 are written out explicitly. They consist of two classes: the short-period terms containing the mean anomaly in the arguments and the long-period terms, containing only the argument of perigee. The two final arguments l'' and g'', the mean anomaly and the argument of the perigee in the notation of Delaunay, are linear with respect to time. Thus, the main effect of the bulge consists in causing the line of apsides and the orbital plane to rotate.

For inclinations smaller than ~ 63.4 the perigee rotates counterclockwise in the orbital plane, assuming the direct motion of the satellite. For inclinations larger than ~ 63.4 the perigee rotates clockwise. The plane itself rotates clockwise for inclinations smaller than $\sim 90^{\circ}$ and it rotates counterclockwise for inclination larger than $\sim 90^{\circ}$. In the closest neighborhood of the critical inclination 63.4 this theory is not applicable.

Kozai (1962) amended Brouwer's solution by inclusion of the periodic terms of order k_2^2. The influence of higher-order zonal harmonics was considered by Giacaglia (1964) and by Garfinkel and McAllister (1964). The problem of inclusion of the tesseral harmonics in the external potential of the earth into Brouwer theory was treated in details by Garfinkel (1965). The general terms in the Hamiltonian is

$$F_{m,\lambda} = - \frac{J_{m,\lambda}}{r^{m+1}} P_m (\sin \theta) \cos \lambda (\varphi - \omega t + \alpha_{m,\lambda})$$

ω designates the angular velocity of the earth's rotation, θ the declination, φ the right ascension of the satellite, $J_{m,\lambda}$ and $\alpha_{m,\lambda}$ are constant. The expansion of $F_{m,\lambda}$ was obtained in terms of the canonical elements of Delaunay. The long periodic terms in this development are of primary importance, especially in the state of resonance with the earth rotations.

The theories of artificial satellites by Sterne (1958), Garfinkel (1958) and Vinti (1959) based on the use of separable Hamiltonians deserve to be mentioned, as well as the theory of Kozai (1959) based on the use of Lagrangian equations of variation of the elliptic elements.

A different approach to the problem was taken by Musen (1959, 1961). He developed a semianalytical theory giving the expansions of Hansen's coordinates into Fourier series with purely numerical coefficients. The basic ideas of Hansen's lunar theory serve the foundation of Musen's theory of artificial satellite: The fundamental idea consists in the introduction of a fictitious auxiliary satellite which moves on the rotating ellipse of constant shape in accordance with Kepler's laws. The position of the real satellite is determined by its deviation from the auxiliary satellite in time and space. The perturbations affecting the satellite are split into the perturbations in the orbit plane and into the perturbations of the orbit plane. The first type of perturbations are the perturbations of the position vector ν and the perturbations of time $n_0\delta z$, which we met already in the exposition of Hansen's lunar theory. They are determined by a single function W. The eccentric anomaly E of the auxiliary satellite is taken as the independent variable. Let Ω be the disturbing function causing the deviation of the satellite from the purely elliptic motion. The basic system of the differential equations determining the perturbations in the orbit plane has the form

$$\frac{dW}{dE} = N\,\Lambda r\,\frac{\partial a_o\Omega}{\partial r} + M\,\Lambda\,\frac{\partial a_o\Omega}{\partial E} + \frac{Sy}{\sqrt{1 - e_o^2}}$$

where

$$(1 - e_o^2)\,M = \frac{h^2}{h_o^2}[-2 + 2e_o\cos E + 2\cos(F - E)$$

$$-e_o\cos(F - 2E) - e_o\cos F]$$

$$+ \frac{1}{1 + \nu}[2e_o^2 - 2e_o\cos E$$

$$+ e_o^2\cos(F + E) + (2 - e_o^2)\cdot$$

$$\cos(F - E) - 2e_o\cos F] - 1 - \tfrac{1}{2}e_o^2$$

$$+ 2e_o\cos E - \tfrac{1}{2}e_o^2\cos 2E$$

$$(1 - e_o^2)\,N = \frac{h^2}{h_o^2}[+ 2e_o\sin E - e_o\sin F$$

$$+ e_o\sin(F - 2E)] + \frac{1}{1 + \nu}\cdot$$

$$\cdot[-2e_o\sin E - (2 - e_o^2)\sin(F - E)$$

$$+ e_o^2\sin(F + E)] + e_o\sin E$$

$$- \tfrac{1}{2}e_o^2\sin 2E$$

$$\Lambda = \frac{1 - \nu^2}{1 + \overline{W}}\left(1 + \frac{y}{\sqrt{1 - e_o^2}}\cdot\frac{\overline{r}}{a_o}\right)$$

$$S = \frac{\rho}{a_o}\cdot\frac{\partial W}{\partial F} - \left(W + 1 + \frac{h_o}{h}\right)e_o\sin F$$

$$\frac{dn_o\delta z}{dE} = \overline{W}\,\frac{r}{a_o} + \left(\frac{\nu^2 - \overline{W}^2}{1 + \overline{W}}\cdot\frac{r}{a_o} - \frac{y}{\sqrt{1 - e_o^2}}\cdot\frac{1 - \nu^2}{1 + \overline{W}}\cdot\frac{\overline{r}^2}{a_o^2}\right)$$

The perturbations of the radius vector are obtained from the equation

$$\overline{W} = -1 - \frac{h_0}{h} + 2\frac{h_0}{h} \cdot \frac{1}{1+\nu}$$

The eccentric anomaly F represents a special device peculiar to Hansen theory. It is introduced to facilitate the integration, as well as to keep the elliptic motion of the auxiliary satellite separated from the perturbations. By h we designated the osculating areal velocity, by h_0 its counterpart associated with the auxiliary ellipse. All elements having the subscript zero are also associated with the auxiliary satellite. After the integration is completed F is replaced by E again. This replacement is represented by the "bar"-operator. The symbol [illegible] represents the speed of rotation of the auxiliary ellipse in its orbit plane. The purely periodic perturbations of the orbit plane are determined by four Euler parameters λ_1, λ_2, λ_3, λ_4. The effects proportional to E are carried by the three basic arguments E by the mean node

$$(\theta) = \theta_0 - (\alpha + \eta)(E - E_0)$$

and by the mean argument of the perigee

$$(\omega) = \omega_0 + (y + \alpha - \eta)(E - E_0)$$

The constants y, α, η appearing in the mean motion of the basic arguments are determined in such a way that no purely secular terms appear in $n_0\delta z$ and in λ_1, λ_2, λ_3, λ_4. As a consequence, the purely secular terms do not appear in the coordinates either.

The theory described takes a slightly simpler form if the true orbital longitude of the satellite is taken as the independent variable instead of E.

The position vector of the satellite is determined by the equations .

$$\vec{r} = A_3[(\theta)] \cdot \Gamma \cdot A_3[(\omega)](1 + \nu) \begin{bmatrix} a_o(\cos E - e_o) \\ a_o\sqrt{1 - e_o} \sin E \end{bmatrix}$$

$$E - e_o \sin E = M_o + n_o(t - t_o) + n_o \delta z$$

where A_3 designates a rotation matrix about the z-axis and Γ is the quaternion-type matrix, rational in parameters λ_1, λ_2, λ_3, λ_4. The main difficulty in understanding Hansen-type theory lies in the fact that it is not expressed in terms of the standard osculating elements. However, with some efforts it is quite lucid. Its strong positive point is that it permits an easy inclusion of the perturbative effects from different sources, such as the influence of the tesseral harmonics, leaving the decision about the relative importance of terms totally to the automatic computer. It is important that care should be taken about the proper numerical accuracy of the geodetic parameters of the earth, because the coefficients of long-period terms are very sensitive to any unnecessary cutting of the numerical values of J_2, J_3, J_4, ..., etc.

For the orbital inclination in the neighborhood of $63^\circ.4$ the mean motion of the argument of the perigee becomes extremely small. The presence of small divisors, however, will cause the terms with the argument g" of Brouwer's theory to become very large in the expressions for the perturbations of the elements. This effect is termed a resonance effect at this particular critical inclination.

Other resonance effects can be caused by tesseral harmonics as well as by the lunar and solar actions. If the period of revolution of the satellite is nearly 24 hours, then the tesseral harmonic associated with the ellipticity of the earth will cause a resonance. Such a satellite nearly follows the earth in its rotation. It can oscillate over two stable locations over the Indian Ocean and Eastern Pacific, or it can drift away from it original position over the earth. In the first approximation the resonance at $63^\circ.4$ resembles a pendulum, i.e., the critical argument g can oscillate between two limits, or to change progressively with time, or to approach a limit asymptotically. The perturbations in the elements are expressible in terms of elliptic integrals of the first and second kind (Hori, 1960). Similar conditions do exist for any type of resonance. Thus the importance of resonance in treating the stability of the orbits is evident. In higher approximations, however, the affinity with the pendulum problem is lost.

The standard technique of developing the perturbations in powers of some small parameter ϵ and in Fourier series fails in the case of a resonance. If the resonant conditions do exist, the development of the perturbations must proceed in fractional powers of the small parameter ϵ, normally in powers of $\epsilon^{\frac{1}{2}}$. The lunar and the solar perturbations can also be of considerable influence on the motion of a close satellite. If the resonance conditions appear, then the lifetime of the satellite can be shortened or prolonged considerably under the influence of the moon, providing the launch time is chosen properly.

The normal way to treat the lunar and solar effects in the motion of a close satellite is based on the elimination of all short-period effects from the corresponding disturbing function. This is equivalent to averaging the disturbing function over the orbit of the artificial satellite. For more distant satellites, this simplified technique is not applicable, because their motion resembles more the motion of a comet of a planet than the motion of a satellite in the strict sense of this word.

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